I BSc 'A' Batch & 'B' Batch

Continuation of Vector Differentiation.

Special Case Problem:-

1) Prove that div $\vec{r} = 3$ and curl $\vec{r} = 0$.

2) Verify that div curl $\vec{F} = 0$ where $\vec{F} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$.

3) Given $\phi = x^3y^2z^4$ prove curl (grad ϕ) $= 0$.

4) If $\vec{V} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is a constant vector then prove $\vec{\omega} = \frac{1}{2}$ curl $\vec{V}$.

5) Prove that curl $(\vec{r} \times \vec{a}) = -2\vec{a}$ where $\vec{a}$ is a constant vector.

6) Prove div $(\vec{r} \times \vec{a}) = 0$ where $\vec{a}$ is a constant vector.

Problem to find ϕ given that $\nabla\phi$:-

1) If $\nabla\phi = 2xyz\,\vec{i} + x^2z\,\vec{j} + x^2y\,\vec{k}$, then find the scalar potential ϕ.

2) Find ϕ, if $\nabla\phi = x(2yz+1)\vec{i} + x^2z\,\vec{j} + x^2y\,\vec{k}$.

3) Let $\vec{F} = (2xy + z^3)\vec{i} + x^2\,\vec{j} + 3xz^2\,\vec{k}$, find ϕ such that $\vec{F} = \nabla\phi$.

4) Find ϕ, if $\nabla\phi = (y + \sin z)\vec{i} + x\,\vec{j} + x\cos z\,\vec{k}$

5) Prove that $\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ is irrotational. Find the scalar potential of $\vec{F}$.

Solution of Special Case Problem:

1) Soln:-

To Prove :- div $\vec{r} = 3$

We know $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{div } \vec{r} = \nabla \cdot \vec{r}$$

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$\nabla \cdot \vec{r} = 3$$

$$\therefore \boxed{\text{div } \vec{r} = 3}$$

To Prove :- curl $\vec{r} = 0$

$$\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= 0 - 0 + 0$$

$$= 0$$

$$\therefore \boxed{\text{curl } \vec{r} = 0}$$

2) Soln:-

To verify:- div curl $\vec{F} = 0$

i.e., $\nabla \cdot \nabla \times \vec{F} = 0$

Given $\vec{F} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$

LHS

$$\nabla \cdot (\nabla \times \vec{F})$$

$$\nabla \times \vec{F} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= \vec{i}\left[2.1z - x.1\right] - \vec{j}\left[0-0\right] + \vec{k}\left[1z - x^2.1\right]$$

$$\nabla \times \vec{F} = \vec{i}\,[2z - x] + \vec{k}\,[z - x^2]$$

Now, $\nabla \cdot (\nabla \times \vec{F})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\,[2z-x] + \vec{k}\,[z-x^2]\right)$$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}(z - x^2)$$

$$= -1 + 1$$

$$= 0$$

$$\therefore \nabla \cdot \nabla \times \vec{F} = 0$$

(or)

div curl $\vec{F} = 0$. //

3) <u>Soln:-</u> To Prove :- Curl (grad ϕ) = 0

$$\nabla \times (\nabla \phi) = 0$$

Given $\phi = x^3 y^2 z^4$

<u>LHS</u>

$$\nabla \times \nabla \phi$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}\left[3x^2 \cdot y^2 z^4\right] + \vec{j}\left[x^3 \cdot 2y\, z^4\right] + \vec{k}\left[x^3 y^2 \cdot 4z^3\right]$$

$$\boxed{\nabla \phi = \vec{i}(3x^2y^2z^4) + \vec{j}(2x^3yz^4) + \vec{k}(4x^3y^2z^3)}$$

Now,

$$\nabla \times \nabla \phi$$

$$= \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2z^4 & 2x^3yz^4 & 4x^3y^2z^3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x^3y^2z^3) - \frac{\partial}{\partial z}(2x^3yz^4)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(4x^3y^2z^3) - \frac{\partial}{\partial z}(3x^2y^2z^4)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(2x^3yz^4) - \frac{\partial}{\partial y}(3x^2y^2z^4)\right]$$

$$= \vec{i}\left[4x^3 \cdot 2y\, z^3 - 2x^3y \cdot 4z^3\right] - \vec{j}\left[4 \cdot 3x^2y^2z^3 - 3x^2y^2 \cdot 4z^3\right]$$

$$+ \vec{k}\left[2 \cdot 3x^2yz^4 - 3x^2 \cdot 2y\, z^4\right]$$

$$= \vec{i}\left[8x^3yz^3 - 8x^3yz^3\right] - \vec{j}\left[12x^2y^2z^3 - 12x^2y^2z^3\right]$$

$$+ \vec{k}\left[6x^2yz^4 - 6x^2yz^4\right]$$

$$= 0$$

$\therefore \nabla \times \nabla \phi = 0$ (or) Curl (grad ϕ) = 0 //

4) Soln:- To Prove:- $\vec{\omega} = \frac{1}{2} \text{Curl } \vec{V}$ (4)

Here $\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$ [Given Constant Vector]

Given $\vec{V} = \vec{\omega} \times \vec{r}$

$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\boxed{\vec{V} = \vec{i}[\omega_2 z - \omega_3 y] - \vec{j}[\omega_1 z - \omega_3 x] + \vec{k}[\omega_1 y - \omega_2 x]}$$

Now. $\text{Curl } \vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (-\omega_1 z + \omega_3 x) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(-\omega_1 z + \omega_3 x)\right]$$
$$- \vec{j}\left[\frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(-\omega_1 z + \omega_3 x) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right].$$

$$= \vec{i}[\omega_1 + \omega_1] - \vec{j}[-\omega_2 - \omega_2] + \vec{k}[\omega_3 + \omega_3]$$

$$\text{Curl } \vec{V} = \vec{i}[2\omega_1] + \vec{j}[2\omega_2] + \vec{k}[2\omega_3]$$

$$= 2[\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}]$$

$$\text{Curl } \vec{V} = 2\vec{\omega}$$

$$\frac{1}{2} \text{Curl } \vec{V} = \vec{\omega}$$

$$\therefore \vec{\omega} = \frac{1}{2} \text{Curl } \vec{V}$$

(5)

5) Soln :- To Prove :- Curl $(\vec{r} \times \vec{a}) = -2\vec{a}$

Here $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ [Given Constant Vector.]

$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r} \times \vec{a} = \begin{vmatrix} \overset{+}{\vec{i}} & \vec{j} & \overset{+}{\vec{k}} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\boxed{\vec{r} \times \vec{a} = \vec{i}\,[a_3 y - a_2 z] - \vec{j}\,[a_3 x - a_1 z] + \vec{k}\,[a_2 x - a_1 y]}$$

Now, LHS Curl $(\vec{r} \times \vec{a}) = \nabla \times (\vec{r} \times \vec{a})$

$$= \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_3 y - a_2 z) & (-a_3 x + a_1 z) & (a_2 x - a_1 y) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(-a_3 x + a_1 z)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_3 y - a_2 z)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-a_3 x + a_1 z) - \frac{\partial}{\partial y}(a_3 y - a_2 z)\right]$$

$$= \vec{i}\,[-a_1 - a_1] - \vec{j}\,[a_2 + a_2] + \vec{k}\,[-a_3 - a_3]$$

$$= \vec{i}\,[-2a_1] - \vec{j}\,[2a_2] + \vec{k}\,[-2a_3]$$

$$= -2\,[a_1\vec{i} + a_2\vec{j} + a_3\vec{k}]$$

Curl $(\vec{r} \times \vec{a}) = -2\vec{a}$.

6) Soln:- To Prove :- $\operatorname{div}(\vec{r}\times\vec{a}) = 0$

Here $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ [Given Constant vector]

$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\vec{r}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\boxed{\vec{r}\times\vec{a} = \vec{i}[a_3y - a_2z] - \vec{j}[a_3x - a_1z] + \vec{k}[a_2x - a_1y]}$$

LHS $\operatorname{div}(\vec{r}\times\vec{a}) = \nabla\cdot(\vec{r}\times\vec{a})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\cdot\left[\vec{i}(a_3y - a_2z) - \vec{j}(a_3x - a_1z) + \vec{k}(a_2x - a_1y)\right]$$

$$= \frac{\partial}{\partial x}(a_3y - a_2z) + \frac{\partial}{\partial y}(-a_3x + a_1z) + \frac{\partial}{\partial z}(a_2x - a_1y)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$\therefore \operatorname{div}(\vec{r}\times\vec{a}) = 0$$

Basic Integral Formula

$$\int \partial x = x$$
$$\int \partial y = y$$
$$\int \partial z = z$$
$$\int \partial \phi = \phi$$
$$\int x\, \partial x = \frac{x^2}{2}$$
$$\int (x+a)\, \partial x = \frac{x^2}{2} + ax$$
$$\int (x+y)\, \partial x = \frac{x^2}{2} + xy$$
$$\int (x+y+z)\, \partial x = \frac{x^2}{2} + xy + xz$$

$$\int y\, \partial y = \frac{y^2}{2}$$
$$\int (x+y)\, \partial y = xy + \frac{y^2}{2}$$
$$\int (x+y+z)\, \partial y = xy + \frac{y^2}{2} + yz$$
$$\int z\, \partial z = \frac{z^2}{2}$$
$$\int (y+z)\, \partial z = yz + \frac{z^2}{2}$$
$$\int (x+y+z)\, \partial z = xz + yz + \frac{z^2}{2}$$

$$\int \sin x\, \partial x = -\cos x$$
$$\int \cos x\, \partial x = \sin x$$

$$\int xyz\, \partial x = \frac{x^2}{2} yz$$
$$\int x^2 y z^2\, \partial x = \frac{x^3}{3} y z^2$$
$$\int x^3 y^4 z^5\, \partial x = \frac{x^4}{4} y^4 z^5$$

$$\int xyz\, \partial z = xy \frac{z^2}{2}$$
$$\int x^2 y z^2\, \partial z = x^2 y \frac{z^3}{3}$$
$$\int x^3 y^4 z^5\, \partial z = x^3 y^4 \frac{z^6}{6}$$

$$\int xyz\, \partial y = x \frac{y^2}{2} z$$
$$\int x y^2 z^2\, \partial y = x \frac{y^3}{3} z^2$$
$$\int x^3 y^4 z^5\, \partial y = x^3 \frac{y^5}{5} z^5$$

$$\int (xy + yz + zx)\, \partial x = \frac{x^2}{2} y + xyz + z\frac{x^2}{2}$$

$$\int (xy + yz + zx)\, \partial y = \frac{xy^2}{2} + \frac{y^2}{2} z + xyz$$

$$\int (xy + yz + zx)\, \partial z = xyz + y\frac{z^2}{2} + \frac{z^2}{2} x.$$

⑦

Solution to find ϕ given that $\nabla\phi$:-

$\boxed{① , ②}$ & ③

Try

3) Soln:- ~~$\nabla\phi = 2xyz$~~

$$\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$$

Given $\vec{F} = \nabla\phi$.

$$\therefore \nabla\phi = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$$

Taking Coeff. of $\vec{i}$,

$$\frac{\partial\phi}{\partial x} = (2xy + z^3)$$

$$\partial\phi = (2xy + z^3)\,\partial x$$

$$\int\partial\phi = \int(2xy + z^3)\,\partial x$$

$$\phi = 2\frac{x^2}{2}y + xz^3$$

$$\phi = x^2y + xz^3 \longrightarrow ①$$

Taking Coeff of $\vec{j}$,

$$\frac{\partial\phi}{\partial y} = x^2$$

$$\partial\phi = x^2\,\partial y$$

$$\int\partial\phi = \int x^2\,\partial y$$

$$\phi = x^2y \longrightarrow ②$$

Taking Coeff. of $\vec{k}$,

$$\frac{\partial\phi}{\partial z} = 3xz^2$$

$$\partial\phi = 3xz^2\,\partial z$$

$$\int\partial\phi = \int 3xz^2\,\partial z$$

$$\phi = 3x\frac{z^3}{3}$$

$$\phi = xz^3 \longrightarrow ③$$

Combining ①, ② & ③,

$$\phi = x^2y + xz^3 + f(x,y,z)$$

(or)

$$\phi = x^2y + xz^3 + c.$$

4) Soln:- $\nabla\phi = (y+\sin z)\vec{i} + x\vec{j} + x\cos z\,\vec{k}$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y+\sin z)\vec{i} + x\vec{j} + x\cos z\,\vec{k}$$

Taking Coeff. of $\vec{i}$,

$$\frac{\partial\phi}{\partial x} = (y+\sin z)$$

$$\partial\phi = (y+\sin z)\partial x$$

$$\int\partial\phi = \int(y+\sin z)\partial x$$

$$\phi = xy + x\sin z \longrightarrow (1)$$

Taking Coeff. of $\vec{j}$,

$$\frac{\partial\phi}{\partial y} = x$$

$$\partial\phi = x\,\partial y$$

$$\int\partial\phi = \int x\,\partial y$$

$$\phi = xy \longrightarrow (2)$$

Taking Coeff. of $\vec{k}$,

$$\frac{\partial\phi}{\partial z} = x\cos z$$

$$\partial\phi = x\cos z\,\partial z$$

$$\int\partial\phi = \int x\cos z\,\partial z$$

$$\phi = x\sin z \longrightarrow (3)$$

Combining (1), (2) & (3),

$$\phi = xy + x\sin z + f(x,y,z)$$

(or)

$$\phi = xy + x\sin z + C$$

5) Soln:- Given $\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$

To Prove :- $\vec{F}$ is irrotational.

$$\nabla\times\vec{F} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+yz) & (4y+zx) & (-6z+xy) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-6z+xy) - \frac{\partial}{\partial z}(4y+zx)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(-6z+xy) - \frac{\partial}{\partial z}(2x+yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(4y+zx) - \frac{\partial}{\partial y}(2x+yz)\right]$$

(9)

$= \vec{i}[x-x] - \vec{j}[y-y] + \vec{k}[z-z]$

$= 0.$

$\therefore \nabla \times \vec{F} = 0$

$\therefore \vec{F}$ is irrotational.

To find scalar potential (ϕ) of $\vec{F}$:-

put $\vec{F} = \nabla\phi$.

$\nabla\phi = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$

$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$

Taking Coeff. of $\vec{i}$,

$\frac{\partial\phi}{\partial x} = (2x+yz)$

$\partial\phi = (2x+yz)\,\partial x$

$\int\partial\phi = \int(2x+yz)\,\partial x$

$\phi = \frac{2x^2}{2} + xyz$

$\phi = x^2 + \cancel{x}yz \longrightarrow (1)$

Taking Coeff. of $\vec{j}$,

$\frac{\partial\phi}{\partial y} = (4y+zx)$

$\partial\phi = (4y+zx)\,\partial y$

$\int\partial\phi = \int(4y+zx)\,\partial y$

$\phi = 4\frac{y^2}{2} + xyz$

$\phi = 2y^2 + xyz \longrightarrow (2)$

Taking Coeff. of $\vec{k}$,

$\frac{\partial\phi}{\partial z} = \cancel{4y} - (6z-xy)$

$\partial\phi = (-6z+xy)\,\partial z$

$\int\partial\phi = \int(-6z+xy)\,\partial z$

$\phi = -6\frac{z^2}{2} + xyz$

$\phi = -3z^2 + xyz \longrightarrow (3)$

Combining (1), (2) & (3),

$\phi = x^2 + 2y^2 - 3z^2 + xyz + f(x,y,z)$

(or)

$\phi = x^2 + 2y^2 - 3z^2 + xyz + c$